The problems of 5^{th} IGO along with their solutions

Advanced Level

Problems:

1. Two circles ω_1, ω_2 intersect each other at points A, B. Let PQ be a common tangent line of these two circles with $P \in \omega_1$ and $Q \in \omega_2$. An arbitrary point X lies on ω_1. Line AX intersects ω_2 for the second time at Y. Point $Y' \neq Y$ lies on ω_2 such that $QY = QY'$. Line $Y'B$ intersects ω_1 for the second time at X'. Prove that $PX = PX'$

2. In acute triangle ABC, $\angle A = 45°$. Points O, H are the circumcenter and the orthocenter of ABC, respectively. D is the foot of altitude from B. Point X is the midpoint of arc AH of the circumcircle of triangle ADH that contains D. Prove that $DX = DO$.

3. Find all possible values of integer $n > 3$ such that there is a convex n-gon in which, each diagonal is the perpendicular bisector of at least one other diagonal.

4. Quadrilateral $ABCD$ is circumscribed around a circle. Diagonals AC, BD are not perpendicular to each other. The angle bisectors of angles between these diagonals, intersect the segments AB, BC, CD and DA at points K, L, M and N. Given that $KLMN$ is cyclic, prove that so is $ABCD$.

5. $ABCD$ is a cyclic quadrilateral. A circle passing through A, B is tangent to segment CD at point E. Another circle passing through C, D is tangent to AB at point F. Point G is the intersection point of AE, DF, and point H is the intersection point of BE, CF. Prove that the incenters of triangles AGF, BHF, CHE, DGE lie on a circle.

Solutions:

1. Two circles ω_1, ω_2 intersect each other at points A, B. Let PQ be a common tangent line of these two circles with $P \in \omega_1$ and $Q \in \omega_2$. An arbitrary point X lies on ω_1. Line AX intersects ω_2 for the second time at Y. Point $Y' \neq Y$ lies on ω_2 such that $QY = QY'$. Line $Y'B$ intersects ω_1 for the second time at X'. Prove that $PX = PX'$

Proposed by Morteza Saghafian

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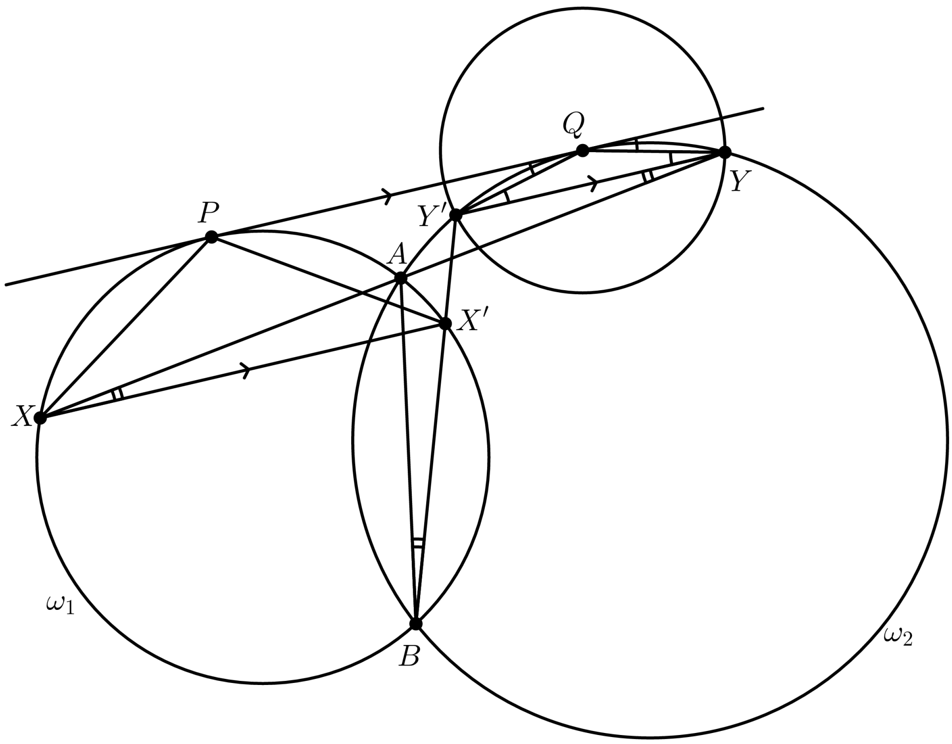
Solution.



$QY = QY'$ implies $\angle QYY' = \angle QY'Y$. Considering circle ω_2, we have $\angle QYY' = \angle Y'QP$. This means $YY' \parallel PQ$.
We also have $\angle Y'YA = \angle Y'BA$ and $\angle ABX' = \angle AXX'$. This means $XX' \parallel YY' \parallel PQ$.
Therefore $\angle PXX' = \angle X'PQ = \angle PX'X$, so $PX = PX'$ ■

2. In acute triangle ABC, $\angle A = 45°$. Points O, H are the circumcenter and the orthocenter of ABC, respectively. D is the foot of altitude from B. Point X is the midpoint of arc AH of the circumcircle of triangle ADH that contains D. Prove that $DX = DO$.

Proposed by Fatemeh Sajadi

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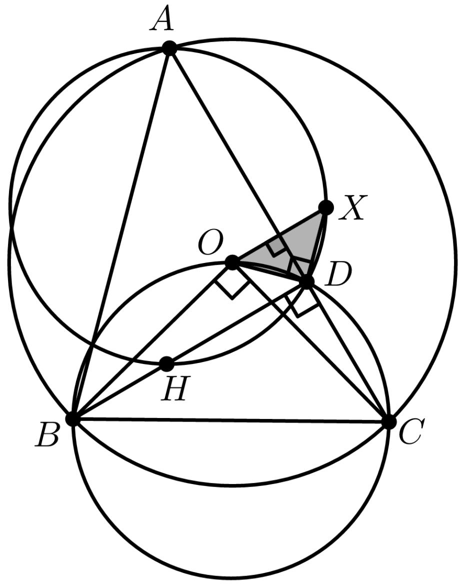
Solution.



Since $\angle AXH = 90°$ and $XA = XH$, we conclude that $\angle AHX = 45° = \angle ADX$.
Also $\angle BOC = 2\angle A = 90°$, therefore points O, D lie on a circle with diameter BC. This implies

$$\widehat{ODA} = \widehat{OBC} = 45° \implies \widehat{ODX} = 90°.$$

But note that

$$\widehat{ACH} = 90° - \widehat{A} = 45° = \frac{1}{2}\widehat{AXH}.$$

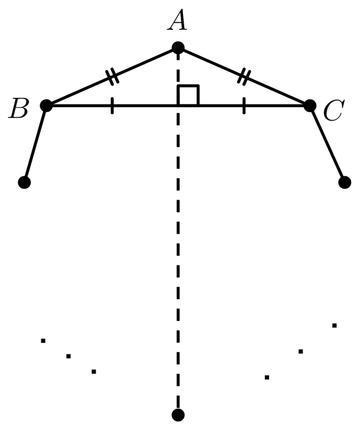
This alongside with $XA = XH$ means X is the circumcenter of triangle ACH and so $XA = XC$. Thus OX is the perpendicular bisector of AC and so $OX \perp AC$. Now in triangle ODX, the angle bisector of vertex D is the same as the altitude from D, hence it is an isosceles triangle with $DX = DO$. ■

3. Find all possible values of integer $n > 3$ such that there is a convex n-gon in which, each diagonal is the perpendicular bisector of at least one other diagonal.

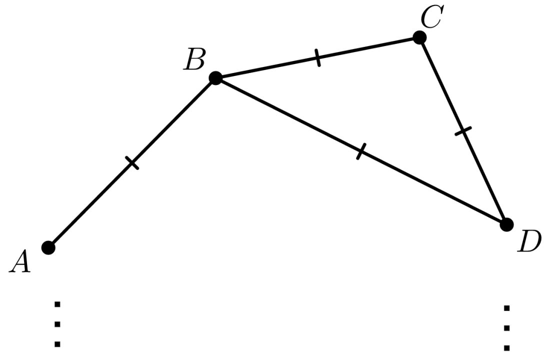
Proposed by Mahdi Etesamifard

Solution. Let m be the total number of the perpendicular bisectors of all diagonals in the given n-gon. The statement of the problem implies that m is not less than the number of diagonals. But it is clear that the total number of perpendicular bisectors of the diagonals does not exceed the number of diagonals! Hence, we conclude that each diagonal is the perpendicular bisector of exactly one other diagonal. Since the perpendicular bisector of a diagonal is a unique line, we get that for each diagonal d, there is exactly one diagonal d' such that d' is the perpendicular bisector of d.

Consider three adjacent vertices B, A, C of the n-gon, where A lies between B and C. BC is a diagonal of the n-gon, and only diagonals that contain A have an intersection point with BC. Specially, the diagonal which is the perpendicular bisector of BC passes through A. Hence $AB = AC$. Using this similar idea, it is deduced that all sides of this n-gon have the same length.



Similar to the previous part, consider four adjacent points of the n-gon, A, B, C, D with the given order. If $n > 4$, then AD is a diagonal of the n-gon, and the only diagonals that contain B or C, have an intersection point with AD. Therefore either B or C lie on the perpendicular bisector of AD. Without loss of generality, assume that $BA = BD$. According to the previous argument, $BA = BC = CD$. Thus triangle BCD is an equilateral and so $\angle BCD = 60°$. (In the other case we would have $\angle ABC = 60°$.)



This implies that between any two adjacent vertices, there is one that has a 60 degree angle. Hence there is at least $\frac{n}{2}$ angles of $60°$ in this n-gon.
It is known that the total number of 60 degree angles in an n-gon with $n > 3$ is at most 2. So we must have $\frac{n}{2} \leq 2$ which means $n \leq 4$, a contradiction.
Clearly, any rhombus satisfies the desired property. So the answer is $\boxed{n = 4}$. ■

4. Quadrilateral $ABCD$ is circumscribed around a circle. Diagonals AC, BD are not perpendicular to each other. The angle bisectors of angles between these diagonals, intersect the segments AB, BC, CD and DA at points K, L, M and N. Given that $KLMN$ is cyclic, prove that so is $ABCD$.

Proposed by Nikolai Beluhov (Bulgaria)

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Solution. Let P be the intersection point of AC, BD. First we claim that KL and MN are not parallel. Assume the contrary, that $KL \parallel MN$. Since $KLMN$ is cyclic, we have $KN = ML$, and $PK = PL,\ PM = PN$. We also have

$$\frac{KP}{PM} = \frac{PL}{PN}.$$

Let $AP = x, BP = y, CP = z$ and $DP = t$. Also let $\angle APB = 2\alpha$ and $\angle BPC = 2\theta$. We have

$$KP = \frac{xy}{x+y}\cos\alpha,\ PM = \frac{zt}{z+t}cos\alpha \implies \frac{KP}{PM} = \frac{\frac{1}{z}+\frac{1}{t}}{\frac{1}{x}+\frac{1}{y}}.$$

Similarly,

$$\frac{PL}{PN} = \frac{\frac{1}{x}+\frac{1}{t}}{\frac{1}{y}+\frac{1}{z}}.$$

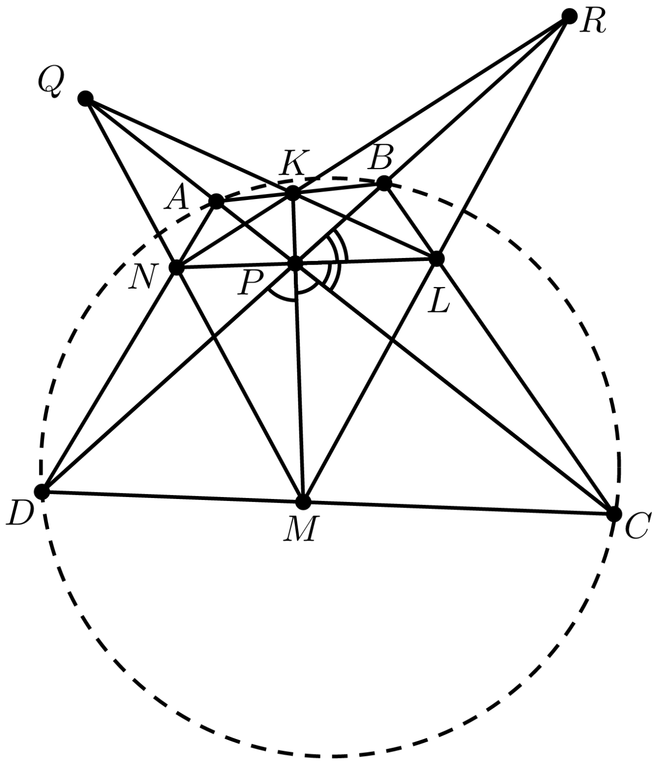
Since $\frac{KP}{PM} = \frac{PL}{PN}$, with a little of calculation, we shall have

$$\frac{1}{yz}+\frac{1}{z^2}+\frac{1}{zt} = \frac{1}{tx}+\frac{1}{x^2}+\frac{1}{xy} \implies \left(\frac{1}{x}-\frac{1}{y}\right)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right) = 0.$$

Which means $x = z$. But note that $PK = PL$ implies

$$\frac{xy}{x+y}\cos\alpha = \frac{yz}{y+z}\cos\theta.$$

But $\theta = 90° - \alpha$, so we must have $\alpha = \theta = 45°$, and so $AC \perp BD$, which is a contradiction. Therefore the claim is proved. With the similar idea, we can show that KN and LM are not parallel.



By Menelaus' theorem, KL and MN meet at a point Q on AC such that $\frac{AQ}{QC} = \frac{AP}{PC}$ and LM and NK meet at a point R on BD such that $\frac{BR}{RD} = \frac{BP}{PD}$.

Let the incircle ω of $ABCD$ touch its sides at K', L', M', and N'. By Brianchon's theorem, AL', CK', and BD are concurrent. By Ceva's and Menelaus' theorems, K', L', and Q are collinear. Analogously, M', N', and Q are collinear and $L'M'$ and $N'K'$ meet at R.

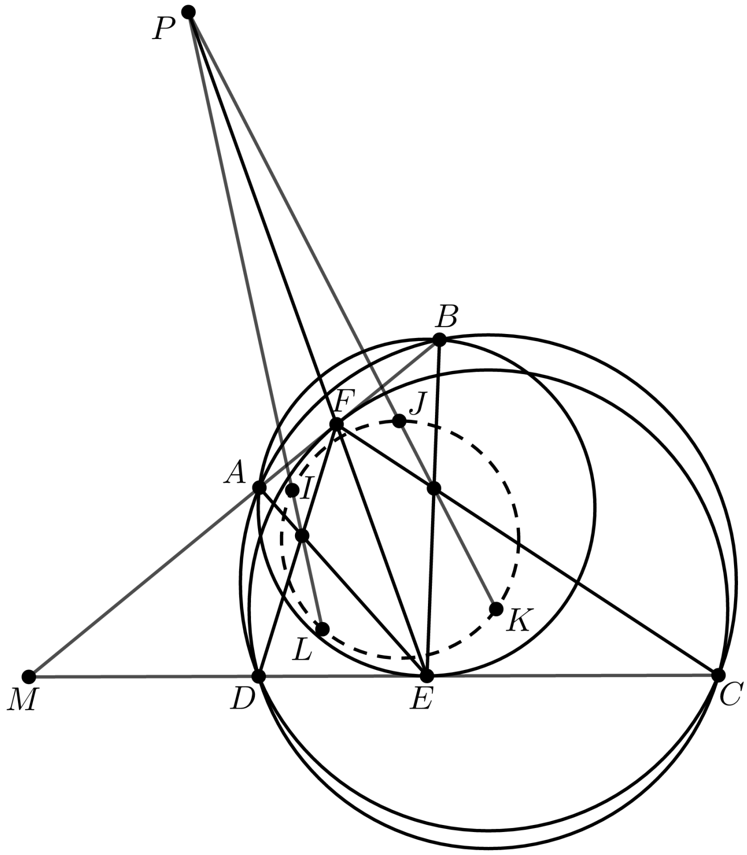
By Brianchon's theorem, $K'M'$ and $L'N'$ meet at P. It follows the diagonals and opposite sides of both $KLMN$ and $K'L'M'N'$ intersect at the vertices of $\triangle PQR$. Therefore, both the circumcircle of $KLMN$ and ω coincide with the polar circle of $\triangle PQR$.

Since K is a common point of AB and ω, $K \equiv K'$. Analogously, $L \equiv L'$, $M \equiv M'$, and $N \equiv N'$. Hence the angle bisector KM of AC and BD makes equal angles with AB and CD and $ABCD$ is cyclic, as needed. ■

5. $ABCD$ is a cyclic quadrilateral. A circle passing through A, B is tangent to segment CD at point E. Another circle passing through C, D is tangent to AB at point F. Point G is the intersection point of AE, DF, and point H is the intersection point of BE, CF. Prove that the incenters of triangles AGF, BHF, CHE, DGE lie on a circle.

Proposed by Le Viet An (Vietnam)

Solution.



Let I, J, K, L be the incenters of the triangles AGF, BHF, CHE, DGE respectively. Let ω be the circumcircle of $ABCD$. In case of $AB \parallel CD$, we would conclude that $ABCD$ is an isosceles trapezoid and it is easy to see that $IJKL$ is also an isosceles trapezoid.
So assume that $AB \nparallel CD$ and let M be the intersection point of rays BA and CD. Since $ABCD$ is cyclic, it is obtained that

$$MA \cdot MB = MD \cdot MC = \mathcal{P}_M(\omega)$$

Since ME is tangent to $\odot ABE$, we get

$$\widehat{MEA} = \widehat{MBE}.$$

We also have $ME^2 = MA \cdot MB = \mathcal{P}_M(\odot ABE)$ and $MF^2 = MD \cdot MC = \mathcal{P}_M(\odot CDF)$, which implies $ME = MF$, and so $\widehat{MEF} = \widehat{MFE}$. Therefore,

$$\widehat{AEF} = \widehat{MEF} - \widehat{MEA} = \widehat{MFE} - \widehat{MBE} = \widehat{BEF}.$$

The latest equation means that EF is the interior angle bisector of $\angle AEB$. Similarly, FE is the interior angle bisector of $\angle CFD$.

Note that H, J, K are collinear and $\angle FJH = 90^\circ + \frac{\angle FBH}{2}$. Thus

$$\begin{aligned}
\widehat{FJK} &= 90^\circ + \frac{\widehat{MBE}}{2} = 90^\circ + \frac{\widehat{MEA}}{2} \\
&= 90^\circ + \frac{180^\circ - \widehat{AEC}}{2} = 180^\circ - \frac{\widehat{AEC}}{2} \\
&= 180^\circ - \frac{\widehat{AEB} + \widehat{BEC}}{2} = 180^\circ - \left(\widehat{FEB} + \widehat{BEK}\right) \\
&= 180^\circ - \widehat{FEK}
\end{aligned}$$

This results in that $EFJK$ is cyclic. With similar arguments, $EFIL$ is also cyclic.
Since EF is the interior angle bisector of $\angle GEH$ and $\angle GFH$, it is easy to see that triangles GEF and HEF are equal. Therefore $EG = EH$ and $FG = FH$, and so $\frac{GE}{GF} = \frac{HE}{HF} = k$. Consider three lines, the exterior angle bisector of vertex G in $\triangle GEF$, the exterior angle bisector of vertex H in $\triangle HEF$ and the line EF. According to the latest equation, there is two cases:

- These three lines are pairwise parallel. This means $EFJK$ and $EFIL$ are isosceles trapezoids. Hence the segments EF, JK and IL have the same perpendicular bisector and so $IJKL$ is an isosceles trapezoid. □
- These three points are concurrent at a point P where $\frac{PE}{PF} = k$. Now we simply have

  $$PJ \cdot PK = \mathcal{P}_P(\odot EFJK) = PE \cdot PF = \mathcal{P}_P(\odot EFIL) = PI \cdot PL.$$

  Which means $IJKL$ is cyclic. □

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